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DIRECT SOLUTION OF THREE—RESERVOIR PROBLEM

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Synopsis

A direct solution for the three-reservoir problem has been presented in this paper. The condition that the upper reservoir may contribute to both the lower reservoirs or that the lowermost reservoir be fed by the other two reservoirs has firstly been obtained. Using this condition the energy equation and the equation of continuity have been solved to obtain a set of diagrams. These diagrams yield a quick and accurate solution for the distribution of discharges.

Introduction

The case of three interconnected reservoirs is met with sometimes in water supply systems. The usual problem in this case is to find the discharges in the three pipes for given elevations of the reservoirs, knowing the length, diameter and the material of the pipe (i.e. the equivalent sandgrain roughness of the pipe). A correct solution of the problem requires consideration of the fact that the Darcy-weisbach resistance coefficient, 'f' of the pipe varies with the Reynolds number and the relative roughness of the pipes; such a procedure has been indicated by Streeter (1). However, for the sake of simplicity a constant value of 'f' may be assumed for each pipe after consideration of the material of the pipe and the likely range of Reynolds number. But the procedure still remains one of trial and error. The usual method is to assume a suitable elevation for the hydraulic gradient at the junction and then solve for the discharges in the three pipes; the correct elevation is the one which satisfies the condition that the inflow into the junction equals the outflow.

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In this paper a method is proposed by which the three reservoir problem can be solved directly without any trial. The assumption made is that the resistance law for each pipe can be expressed as

$$h_f = r\,Q^2$$

where h_f is the head loss in the pipe (including minor losses, if any) at a discharge Q and r is a constant for the given pipe. It can be seen that r can be calculated for any pipeline if the length, L, diameter, D, are known and the Darcyweisbach resistance coefficient 'f' and the minor losses can be estimated.

Conditions governing flow directions in the three pipes

Figure 1 shows a three-reservoir system in which z_1 and z_2 are the elevations of the water surfaces of the first and second reservoirs above the water surface of the lowermost reservoir. Let the elevation of the hydraulic gradient at the junction above the surface of the lowermost reservoir be z_j. It is obvious that flow takes place from Reservoir 1 towards the junction and from the junction towards the lowermost reservoir. But the direction of flow in the second pipe is not so obvious. In fact if the direction of flow in the second pipe is known, the problem can be solved directly without any trials. The direction of flow in the second pipe can be determined as follows: Let us assume that $z_j = z_2$. Then obviously Q_2, the discharge in the second pipe, is zero. Also

$$Q_1 = \sqrt{\frac{z_1 - z_2}{r_1}} \qquad (2)$$

and

$$Q_3 = \sqrt{\frac{z_2}{r_3}} \qquad (3)$$

where Q_1, and Q_3 are the dischanges in the pipes 1 and 3 respectively and r_1 and r_3 are their respective resistance parameters. Obviously the assumed elevation for z_j would be correct only if

$$\sqrt{\frac{z_1 - z_2}{r_1}} = \sqrt{\frac{z_2}{r_3}} \qquad (4)$$

In that case Q_1 and Q_3 can be computed from Eq (2) and (3) and $Q_2 = 0$.

If, however, $\sqrt{\frac{z_1 - z_2}{r_1}} > \sqrt{\frac{z_2}{r_3}}$,

the value of z_j will have to be increased from its trial value of z_2 to satisfy the continuity equation for the junction. This means that flow takes place from the junction towards the reservoir in pipe 2. The necessary conditon for this can be simplified as shown below.

$$\frac{z_1 - z_2}{r_1} > \frac{z_2}{r_3}$$

$$\frac{z_1 - z_2}{z_2} > \frac{r_1}{r_3}$$

$$x - 1 > R_1$$

where $x = z_1/z_2$ and $R_1 = r_1/r_3$

$$\therefore \quad x - 1 - R_1 > 0 \qquad (5)$$

forms the condition that flow takes place from the junction to the reservoir in pipe 2.

Similarly it can be shown that

$$x - 1 - R_1 < 0 \qquad (6)$$

for flow to take place from the reservoir to the junction in pipe 2. Thus the problems of three interconnected reservoirs can be classed under the following categories.

Case I: $x - 1 - R_1 > 0$

The corresponding continuity equation becomes $Q_1 = Q_2 + Q_3$ (7)

Case II: $x - 1 - R_1 = 0$

Then $Q_2 = 0$

and $Q_1 = Q_3 = \sqrt{\frac{z_1 - z_2}{r_1}}$

Case III: $x - 1 - R_1 < 0$

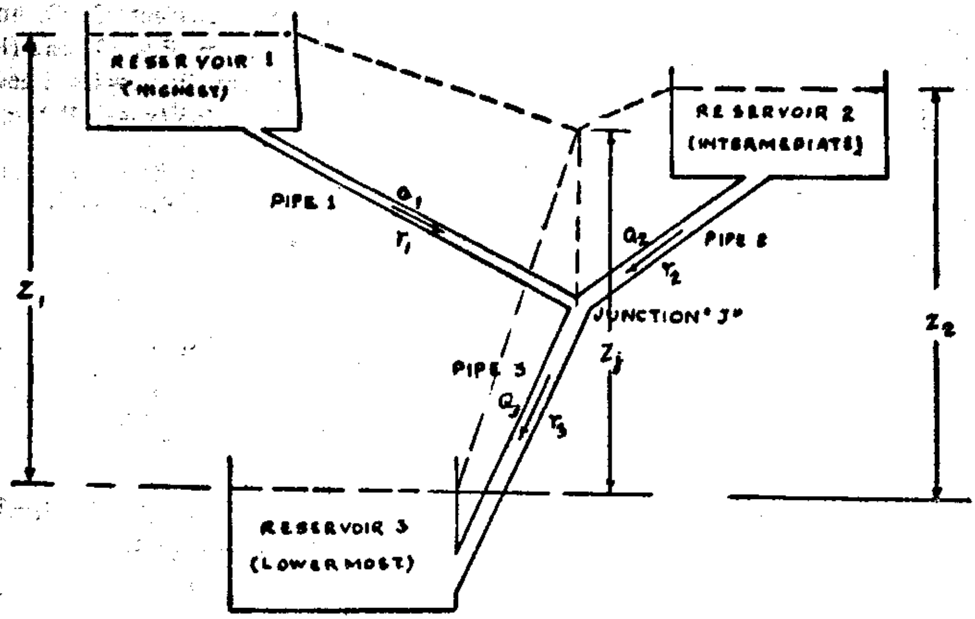

FIG.1 DEFINITION SKETCH

The continuity equation for this case is $Q_1+Q_2=Q_3$ (3)

Thus determination of the value of $(x-1-R_1)$ enables classification of the particular problem under the above heads and the solution for case II is seen to be extremely simple. The solution for cases I and III is discussed below.

Case I: Flow from Reservoir 1 to Reservoirs 2 and 3: (condition $x-1-R_1>0$)

The equations for head loss and the continuity equation for this can be expressed as follows;

$$Q_1 = Q_2 + Q_3 \quad (7)$$

$$z_1 = r_1 Q_1^2 + r_3 Q_3^2 \quad (9)$$

$$z_1 - z_2 = r_1 Q_1^2 + r_2 Q_2^2 \quad (10)$$

$$\therefore 1 - \frac{z_2}{z_1} = \frac{r_1 Q_1^2 + r_2 Q_2^2}{r_1 Q_1^2 + r_3 Q_3^2}$$

Putting $Q_2/Q_1 = y$ and $Q_3/Q_1 = 1-y$, we get

$$1 - \frac{1}{x} = \frac{R_1 + R_2 y^2}{R_1 + (1-y)^2}, \text{ where } R_2 = r_2/r_3.$$

Simplifying,

$$(R_2 x + 1 - x) y^2 + 2y(x-1) - (x - R_1 - 1) = 0$$

Solving the above equation,

$$y = -\frac{x-1}{R_2 x + 1 - x} \pm \sqrt{\left(\frac{x-1}{R_2 x + 1 - x}\right)^2 + \frac{x - R_1 - 1}{R_2 x + 1 - x}} \quad (11)$$

The above equation yields two values of y and it needs to be examined as to which of these corresponds to the physical case under consideration. In this particular case $(x-1)$ and $(x-R_1-1)$ are both positive but $(R_2 x + 1 - x)$ can be positive or negative depending on the value of R_2. When (R_2x+1-x) is positive, the equation gives one positive and one negative value of y, the latter value being meaningless. But when (R_2x+1-x) is negative, the equation gives two positive values of y. It can be easily shown that one of these roots will be less than 1.0, while the other will be greater than 1.0. Since in case I the value of y can only lie between 0 and 1.0, the required value of y is that which is less than 1.0. Thus for both positive and negative values of (R_2x+1-x), there is only one possible value of y and thus a unique discharge distribution in the three pipes.

Equation (11) can be written as

$$y=-B \pm \sqrt{B^2+A} \tag{12}$$

$$\text{where } B = \frac{x-1}{R_2x+1-x} \tag{13}$$

$$\text{and } A = \frac{x-R_1-1}{R_2x+1-x} \tag{14}$$

$$\text{or} \quad B^2+y^2+2By = B^2+A$$

$$\therefore \quad A = y^2+2By \tag{15}$$

Equation (15) was solved to obtain A for different values of y and B. It should be noticed, that in any particular problem, A and B will have the same sign (both positive or both negative) since $(x-1)$ and $(x-R_1-1)$ are always positive. This condition was ensured in choosing the combination of values of y and B in solving Eq. 15. The range of y chosen in these calculations is from 0.05 to 0.9 and the range of the parameter B was decided by considering the extreme values of x, R_1 and R_2 from practical considerations. The results of these calculations are shown on Fig. 2. Since the parameters A and B are known from the given data, the value of y can be read from Fig. 2. Using this value of y, the discharges Q_1, Q_2 and Q_3 can be obtained from Eqs. (9) and (10).

Case III : Flow from Reservoirs 1 and 2 to Reservoir 3 : (Condition $(x-1-R_1) < 0$)

The equations for head loss and the continuity equation for this case are

$$Q_1 + Q_2 = Q_3 \tag{8}$$

$$z_1 = r_1 Q_1^2 + r_3 Q_3^2 \tag{16}$$

$$z_2 = r_2 Q_2^2 + r_3 Q_3^2 \tag{17}$$

From Eqs. (16) and (17)

$$x = \frac{R_1 + (1+y)^2}{R_2 y^2 + (1+y)^2}$$

Simplifying,

$$(R_2 x+x-1) y^2+2y (x-1)+(x-R_1-1)=0$$

Solving the above equation

$$y = -\frac{x-1}{R_2 x+x-1} \pm \sqrt{\left(\frac{x-1}{R_2 x+x-1}\right)^2 - \frac{x-R_1-1}{R_2x+x-1}} \tag{18}$$

It should be noticed that in this case, $(x-1)$ and $(R_2 x+x-1)$ are always positive and $(x-R_1-1)$ is always negative. Thus Eq. (18) will always yield a positive and a negative value of y, the latter value being meaningless. This indicates again a unique discharge distribution in the system.

Eq. (18) can be written as

$$y = -D+\sqrt{D^2+C} \tag{19}$$

$$\text{where} \quad D=\frac{x-1}{R_2x+x-1} \quad \text{and}$$

$$C= -\frac{(x-R_1-1)}{R_2x+x-1}$$

$$\therefore \quad y^2+D^2+2y D = D^2+C$$

$$\text{or} \quad C = y (y^2+2y D) \tag{20}$$

Equation (20) was solved to obtain values of C for various values of y and D. The value of y was varied from 0.05 to 20.0 in these computations. Considering practically extreme values of x, R_1 and R_2, range of the parameter D was fixed. The

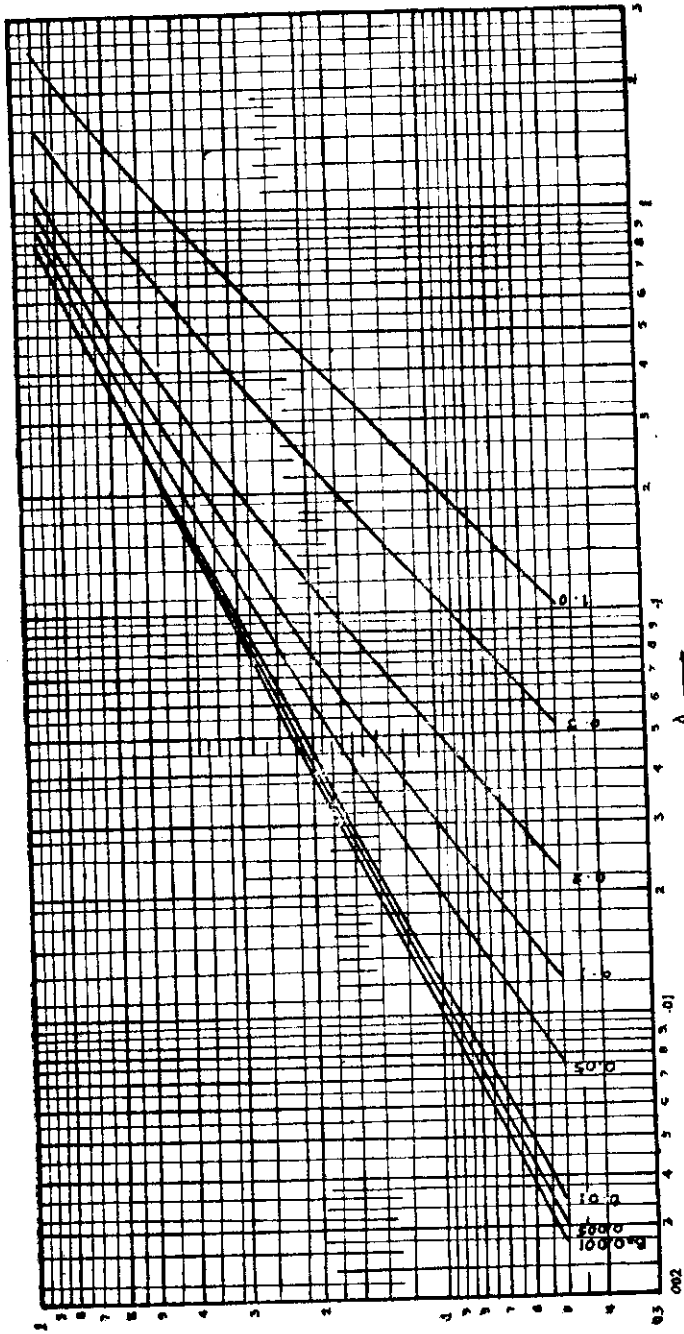

Fig. 2 (a) Relation Between "A" and "Y" For Various Values of "B"

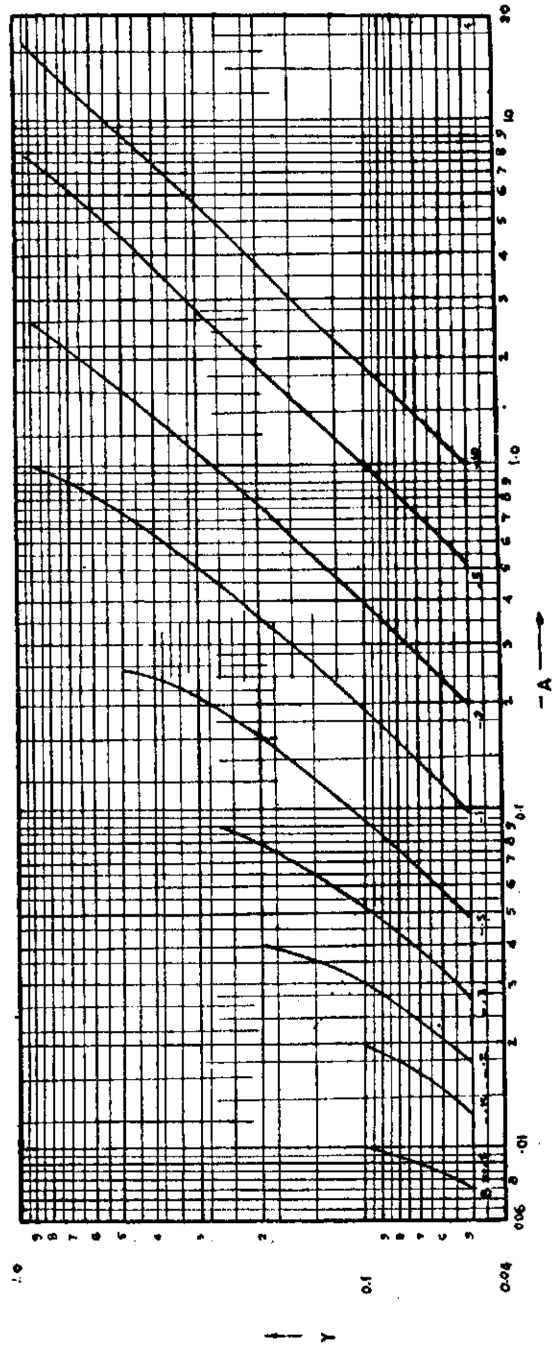

FIG.2 b. RELATION BETWEEN "-A" AND "Y" FOR VARIOUS VALUES OF b.

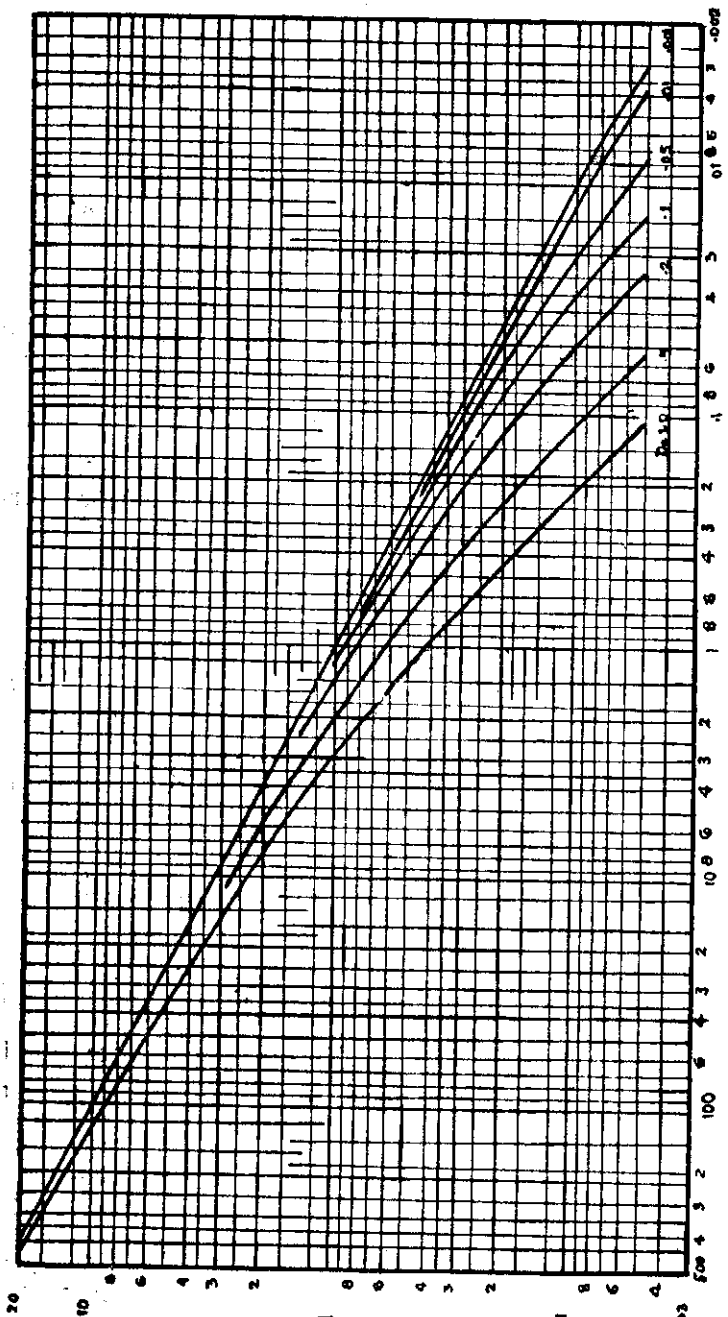

FIG. 3 RELATION BETWEEN "C" AND "Y" FOR DIFFERENT VALUES OF "D"

results of these calculations are shown in Fig. 3. The value of y can be read from this figure for known values of C and D; equations (16) and (17) can then be used to obtain the values of Q_1, Q_2 and Q_3.

Two typical problems are solved below to illustrate the method of using Figs. 2 and 3.

Prob. 1

To find the discharges in the branching system from the following data (Minor losses neglected) :

Water surface in reservoir 1 is 20m above that in reservoir 3

Water surface in reservoir 2 is 10 m above that in reservoir 3.

Pipe 1 is 100 cm. dia, 3000 m long with f = .015

Pipe 2 is 50 cm. dia, 600 m long with f = .024

Pipe 3 is 60 cm. dia, 1200 m long with f = .02

Solution :

From the above data,

$$r_1 = \frac{8\, f_1\, L_1}{\pi^2 \times (D_1)^5 \times g} = \frac{1 \times .015 \times 3000}{\pi^2 \times 1 \times 9.81} = 3.72$$

Similarly, $r_2 = 38.2$

$r_3 = 25.5$

$x = z_1/z_2 = 20/10 = 2$; $R_1 = r_1/r_3 = 0.146$; $R_2 = r_2/r_3 = 1.5$; $x - 1 - R_1 = 2 - 1 - 0.146 > 0$.

Hence this problem falls under case I.

$$\text{Now,}\quad A = \frac{x - R_1 - 1}{R_2\, x + 1 - x} = 0.427$$

$$B = \frac{x - 1}{R_2\, x + 1 - x} = 0.5$$

Referring Fig. (2a) for the above values of A and B, $y = 0{,}32$

With this value of y in equations (10) and (7),

$Q_1 = 1.145\ m^3/sec.$

$Q_2 = 0.366\ m^3/sec$ and

$Q_3 = 0.779\ m^3/sec$

These values agree very closely with those computed by the trial and error method.

Prob. 2

To find the discharges in the branching system with the following data for pipe lines :

Water surface in reservoir 3 is 30 m below that in reservoir 1 and 20m below that in reservoir 2.

Pipe line 1 from reservoir 1 —1000 m long and 10 cm. dia.

Pipe line 2 from reservoir 2 —50 m long and 10 cm dia.

Pipe line 3 from reservoir 3 —100 m long and 10 cm. dia.

Friction factor, "f" for all the three pipes = 0.02.

Solution

From the given data

$r_1 = 1.65 \times 10^5$

$r_2 = 8.3 \times 10^3$

$r_3 = 1.65 \times 10^4$

$R_1 = 10$, $R_2 = 0.5$ and $x - 1 - R_1 = 1{,}3 - 1 - 0.38 < 0$; thus the problem falls under case III

$$\text{Further } D = \frac{x - 1}{R_2\, x + x - 1} = 0.4 \text{ and}$$

$$C = -\frac{x - R_1 - 1}{R_2\, x + x - 1} = 7.6$$

Referring Fig. (3) for the above values of C and D,

$$y = 2.3 = Q_2/Q_1$$

$$\therefore \quad Q_3/Q_1 = 3.3.$$

Substitution in (8) and (16) yields

$Q_1 = 0.931 \times 10^{-2}$ m³/sec.

$Q_2 = 2.14 \times 10^{-2}$,,

$Q_3 = 3.07 \times 10^{-2}$,,

The values computed by the trial and error method are very close to the above values.

Conclusions

The solution to the three reservoir problem has been put in the form of a set of diagrams. These diagrams enable direct solution of the problem avoiding the tedious process of trial and error.

Reference

1. STREETER, V.L., "*Fluid Mechanics,*" McGraw Hill Book Co, Inc 1962.

Notations

h_f = Head loss in pipe (including minor losses, if any).
Q = Discharge flowing through pipe.
r = Resistance parameter for the pipe.
R = Dimensionless relative resistance parameter.
L = Length of the pipe line.
D_1 = Diameter of the pipe.
f = Darcy-Weisbach coefficient.
z = Elevation of the water surfaces in the reservoirs above that in the lowest.
z_j = Elevation of the hydraulic gradient at the junction above the water surface of the lowermost reservoir.
1,2,3 = Subscripts to refer the appropriate pipe line and the reservoir.
g = Acceleration due to gravity
A,B,C,D = Dimensionless parameters introduced in the analysis.
x = z_1/z_2